

# RANDOMNESS IN NUCLEI AND IN THE QUARK-GLUON PLASMA

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The issue of averaging randomness is addressed, mostly in nuclear physics, but shortly also in QCD. The Feshbach approach, so successful in dealing with the continuum spectrum of the atomic nuclei (*optical model*), is extended to encompass bound states as well (*shell model*). Its relationship with the random-matrix theory is discussed and the bearing of the latter on QCD, especially in connection with the spectrum of the Dirac operator, is briefly touched upon. Finally the question of whether Feshbach's theory can cope with the averaging required by QCD is considered.

## 1 The random-matrix theory and nuclear applications

The random-matrix theory (RMT) is built out of any stochastic modelling of the matrix representation of a Hamiltonian. It was introduced half-a-century ago by Wigner, who considered an ensemble of Hamiltonians, different but sharing some symmetry, having independent random variables as elements and searched for those properties common to (nearly) all the members of the ensemble. For this purpose he introduced the probability of the occurrence of a given matrix  $H$  in the ensemble by multiplying a weighting function  $P_{N\beta}(H)$ , —  $N$  being the Hilbert space dimension and  $\beta$  an index specifying the symmetry of the ensemble, — for the differentials of all the elements of the matrix representing  $H$ .

Wigner focused on matrices with random elements having a gaussian distribution (Gaussian statistical hypothesis). The weighting function of a Hamiltonian with  $N$  quantum states is then

$$P_{N\beta}(H) = e^{-\beta N \text{tr} H^2 / \lambda^2}, \quad (1)$$

$\lambda$  being some  $N$ -independent constant.

Concerning the symmetry, in the framework of the quantum Gaussian random-matrix theory (GRMT) one can define three different ensembles:

- i) the ensemble of the real symmetric matrices, describing time-reversal and rotationally invariant systems, referred to as GOE (Gaussian orthogonal ensemble) and corresponding to  $\beta = 1$ ;

- ii) the ensemble of Hermitian matrices, describing systems violating time-reversal invariance, as, e. g., a nucleus in an external magnetic field, referred to as GUE (Gaussian unitary ensemble) and characterized by  $\beta = 2$ ;
- iii) the ensemble of the matrices that are linear combinations of the 2x2 unit matrix and the Pauli matrices, namely

$$H_{nm} = H_{nm}^{(0)} \mathbf{1}_2 - i \sum_{j=1}^3 H_{nm}^{(j)} \sigma_j, \quad (2)$$

where  $H_{nm}^{(0)}$  and  $H_{nm}^{(j)}$  are real matrices, symmetric and antisymmetric, respectively. The matrices (2) describe systems with half-integer spin, time-reversal, but not rotationally, invariant and the associated ensemble is referred to as GSE (Gaussian symplectic ensemble), which corresponds to the choice  $\beta = 4$ .

The elements of the matrices belonging to the ensembles GOE, GUE and GSE are real, complex and quaternion numbers, respectively.

How to derive predictions on observables and how to compare these with the data in the GRMT framework are questions recently reviewed by Weidenmueller *et al.*<sup>1</sup> These authors emphasize that predictions having universal validity, i. e. unrelated to the specific nature of the system under investigation, concern the fluctuations around mean values. Typical in this connection is the energy spectrum of a complex system with its mean level spacing  $\Delta(\epsilon)$  and the related fluctuations: In the GRMT the latter are universal, the energies of the individual systems are not.

More generally, are the local correlations among the eigenvalues (and the eigenvectors as well) that are accounted for by the GRMT, using  $\Delta(\epsilon)$  as an input to fix the parameter  $N/\lambda^2$  in (1). As it is well-known, an amazing example in this connection is offered by the short-range fluctuations in the spectra of the atomic nuclei near the neutron emission threshold ( $\sim 8$  MeV). Indeed, given  $s = S/\Delta(\epsilon)$  where  $S$  denotes the actual level spacing, the data of the nearest neighbour spacing distributions of the nuclear levels,  $p(s)$ , are strikingly in accord with the GRMT predictions as shown in Fig. 1.

It is of importance to realize that the strong repulsion displayed by the nuclear levels, when their excitation energy is sufficiently large and their distance sufficiently small, in the GRMT is engrained in Eq. (1). Indeed, if the eigenvalues  $E_1, E_2, \dots, E_N$  and the eigenvectors of the matrices of the ensemble are chosen as new independent variables, then (1) factorizes in two terms:

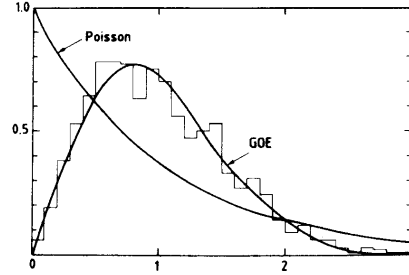


Figure 1. Nearest-neighbor spacing distribution versus the variable  $s$  defined in the text. The GRMT prediction (denoted GOE) is compared to the data (histogram). Also shown the result obtained with a Poisson distribution. (Taken from Ref. <sup>1</sup>).

One only eigenvalue, the other only eigenvector dependent. The former reads

$$P_{N\beta}(E_1..E_N) = \prod_{m>n} |E_m - E_n|^\beta \prod_{l=1}^N dE_l \quad (3)$$

and clearly displays the levels repulsion in the Vandermonde determinant that multiplies  $dE_1..dE_N$ .

Actually, there is still a long way to go from (3) to  $p(s)$ . It is remarkable that Wigner was able to cross it with his famous ansatz

$$p(s) = ase^{-bs^2}, \quad (4)$$

which turned out to be very close to the exact predictions of the GRMT.<sup>2</sup> Note that the gaussian fall-off of (4) is not related to the gaussian nature of the ensemble, but directly arises from the Vandermonde determinant in (3).

Applications of GRMT to QCD will be just touched upon in Sec. 3.

## 2 Feshbach's theory

Feshbach's approach assumes the existence of randomness in complex systems whose internal structure is revealed by their excitation spectrum. On the other hand, in this view there are no systems completely ruled by randomness. The latter is defined through the relation

$$\langle \sigma(A)\sigma(B) \rangle = \langle \sigma(A) \rangle \langle \sigma(B) \rangle, \quad (5)$$

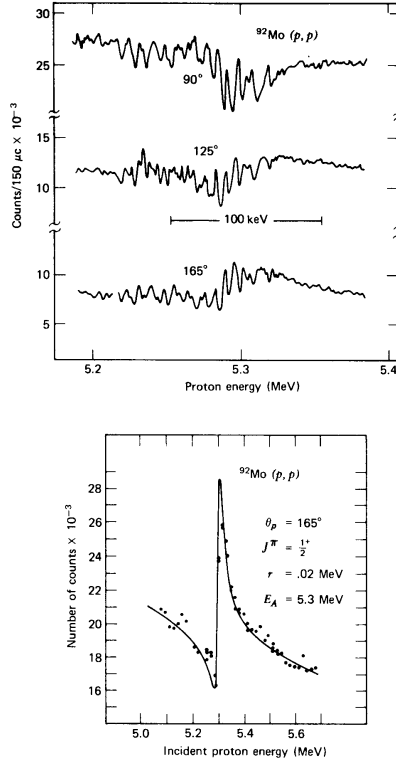


Figure 2. (a) Elastic scattering of protons by  $^{92}\text{Mo}$  for energies close to the  $s$ -wave isobar analog resonance; (b) what one obtains by averaging the data of (a). (Taken from Ref. <sup>4</sup>).

$\sigma$  being some physical observable of the system, dependent upon a parameter: The system is chaotic if the value of  $\sigma$  in correspondence of the value  $A$  for the parameter is independent of the value  $\sigma$  assumes for any other value of the parameter, say  $B$ .

Feshbach's <sup>3</sup> view is that systems totally ruled by randomness do not exist or, equivalently, that coherent combinations of  $\sigma$  can always be found such to allow the existence of correlations where now, in contrast with the discussion of Sec. 1, the correlations are *dynamical*, not *statistical*, they relate to *order*, not to *disorder*.

The question then arises: How these dynamical correlations can be unraveled? The answer is: With appropriate averaging procedures. Before formally

defining the average, it is illuminating to experience how it works in a specific example: The elastic scattering of protons from the nucleus  $^{92}\text{Mo}$  at various scattering angles and in the energy range of a few MeV. In Fig. 2a we show the data obtained with a poor energy resolution: They indeed appear to display the features of a random process (note that the values  $A$  and  $B$  of a generic parameter in (5) here correspond to specific values of the energy). But in Fig. 2b the same data (at  $\theta = 165^\circ$ ) are shown as obtained with a high energy resolution. What appears is astonishing: A perfect resonance typical of a highly correlated scattering process.

The Heisenberg uncertainty principle  $\Delta E \Delta t \geq \hbar/2$ , provides the key to understand. Poor energy resolution means large  $\Delta E$ , hence small  $\Delta t$  or short time for the system to interact: The data appear random. On the other hand, small  $\Delta E$  entails large  $\Delta t$ , hence a long time for the system to average out the fluctuations: The results are then those of Fig. 2b.

The necessity of averaging procedures to properly deal with randomness is thus clear. This program has been carried out by Feshbach<sup>5</sup> in his unified theory of nuclear reactions. We here extend his formalism to the bound states of many-body systems, focusing on the ground state, to complete Feshbach's program and to illustrate how it works. For this purpose we start by partitioning the Hilbert space associated to a many-body system in the  $P$  and  $Q$  sectors, respectively.

### 2.1 Theoretical framework

As it is well-known, the splitting of the Hilbert space induced by the projection operators  $P$  and  $Q$  transforms the Schroedinger equation

$$H\psi = E\psi \quad (6)$$

into the pair of coupled equations

$$(E - H_{PP})(P\psi) = H_{PQ}(Q\psi) \quad (7)$$

$$(E - H_{QQ})(Q\psi) = H_{QP}(P\psi), \quad (8)$$

the meaning of the symbols being obvious. From the above the equation obeyed solely by  $(P\psi)$  is derived. It reads

$$\mathcal{H}(P\psi) = E(P\psi), \quad (9)$$

the  $P$ -space Hamiltonian being

$$\mathcal{H} = H_{PP} + H_{PQ} \frac{1}{\left(\frac{1}{e_Q}\right)^{-1} + W_{QQ}} H_{QP}, \quad (10)$$

with

$$e_Q = E - H_{QQ} - W_{QQ} \quad (11)$$

$$W_{QQ} = H_{QP} \frac{1}{E - H_{PP}} H_{PQ}. \quad (12)$$

It is of significance that although Eq. (9) is not an eigenvalue equation, since the energy  $E$  also appears in the denominator of its right hand side, yet its solutions only occur for those values of  $E$  which are eigenvalues of (6) as well; in other words, a one-to-one correspondence between the values of  $E$  allowed by (6) and (9) exists (see later for a further discussion of this point). However, the solutions of (9) in correspondence to the various values of  $E$  are *not* orthogonal.

Now, we assume the quantum deterministic aspect of nuclear dynamics to be embodied in the  $P$ -space, the chaotic one in the  $Q$ -space. Hence, the strategy of averaging over the latter follows, although, admittedly, some fuzziness does affect this partitioning. To set up the averaging procedure we start by the recognition that the wave functions in the  $Q$ -space are rapidly varying functions of the energy  $E$ , viewed as a parameter classifying their ensemble.

Accordingly, we average over this ensemble following the prescription

$$\langle f(E) \rangle = \int_{-\infty}^{\infty} dE \rho(E, \bar{E}_0, \epsilon) f(E), \quad (13)$$

$f$  being a generic function to be averaged over the variable  $E$  with the distribution  $\rho(E, \bar{E}_0, \epsilon)$ . The latter depends, beyond  $E$ , also upon the value  $\bar{E}_0$  around which the average, — taken over a range of  $E$  essentially set by  $\epsilon$ , — is performed. A distribution convenient for our purposes is

$$\rho(E, \bar{E}_0, \epsilon) = \frac{1}{2\pi i} \frac{e^{iE\eta}}{E - (\bar{E}_0 - \epsilon) - i\eta}, \quad (14)$$

which is indeed correctly normalized being

$$\int_{-\infty}^{\infty} dE \rho(E, \bar{E}_0, \epsilon) = 1 \quad (15)$$

(one should let  $\eta \rightarrow 0^+$  after the integration has been performed). Note that Eq. (14) extends in some sense the Lorentz distribution of the optical model<sup>5</sup> to the situation of a zero width state. Hence the present formalism is especially suited to deal with ground states, which are of course stable: We shall accordingly focus mainly on the latter in the following.

Now, in the  $Q$ -space the wave functions are found to be<sup>6,7,8</sup>

$$(Q\psi) = \frac{1}{e_Q} H_{QP} \psi_0, \quad (16)$$

$\psi_0$  being an auxiliary function that in the end disappears from the formalism. By averaging Eq. (16) according to the prescriptions (13) and (14), one then finds that the averaged wave function of the nuclear ground state in the  $P$ -space (here denoted by the angle brackets) obeys the equation

$$\bar{\mathcal{H}}\langle P\psi\rangle = \bar{E}_0\langle P\psi\rangle. \quad (17)$$

In (17)  $\bar{E}_0$  is the mean field energy and

$$\bar{\mathcal{H}} = H_{PP} + H_{PQ} \frac{1}{\left\langle \frac{1}{e_Q} \right\rangle^{-1} + W_{QQ}(E = \bar{E}_0)} H_{QP} \quad (18)$$

is the mean field Hamiltonian. This can be further elaborated since the singularities of the operator  $1/e_Q$  lie in the  $\text{Im}E < 0$  half-plane<sup>9</sup>. Accordingly, one gets

$$\begin{aligned} \left\langle \frac{1}{e_Q} \right\rangle &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dE \frac{e^{iE\eta}}{E - (\bar{E}_0 - \epsilon) - i\eta} \frac{1}{E - H_{QQ} - W_{QQ}(E)} \\ &= \frac{1}{\bar{E}_0 - \epsilon - H_{QQ} - W_{QQ}(E = \bar{E}_0 - \epsilon)} \approx \frac{1}{\bar{E}_0 - \epsilon - H_{QQ} - W_{QQ}(E = \bar{E}_0)}, \end{aligned} \quad (19)$$

the last passage holding if the energy dependence of the operator  $W_{QQ}$  is mild and if the parameter  $\epsilon$  is not too large (it should be not too small either, otherwise the energy averaging procedure becomes meaningless).

The insertion of (19) into (18) leads then to the following useful alternative expression for the mean field Hamiltonian

$$\bar{\mathcal{H}} = H_{PP} + V_{PQ} V_{QP} \frac{1}{\bar{E}_0 - \epsilon - E}, \quad (20)$$

where the energy dependent operators

$$V_{PQ} = H_{PQ} \sqrt{\frac{\bar{E}_0 - \epsilon - E}{\bar{E}_0 - \epsilon - H_{QQ}}} \quad (21)$$

and

$$V_{QP} = \sqrt{\frac{\bar{E}_0 - \epsilon - E}{\bar{E}_0 - \epsilon - H_{QQ}}} H_{QP}, \quad (22)$$

represent the residual effective NN interaction. The usefulness of the Eqs. (20), (21) and (22) was realized in Ref. <sup>10</sup>, where it was noticed that with their help

the pair of equations (2.1) can be recast, as far as  $(P\psi)$  is concerned, into the form

$$(E - \bar{\mathcal{H}})(P\psi) = V_{PQ}(Q\psi) \quad (23)$$

$$(E - H_{QQ})(Q\psi) = V_{QP}(P\psi), \quad (24)$$

which is suitable for expressing the mean field fluctuations (the “error”).

Indeed, by exploiting the spectral decomposition of the operator  $(E - \bar{\mathcal{H}})^{-1}$  in terms of the eigenfunctions  $\phi_n$  of the mean field Hamiltonian  $\bar{\mathcal{H}}$ ,<sup>a</sup> one gets from Eq. (23)

$$\begin{aligned} |P\psi\rangle &= \sum_n \frac{|\phi_n\rangle}{E - \bar{E}_n} \langle\phi_n|V_{PQ}|Q\psi\rangle \\ &= |\phi_0\rangle \frac{\langle\phi_0|V_{PQ}|Q\psi\rangle}{E - \bar{E}_0} + \left(\frac{1}{E - \bar{\mathcal{H}}}\right)' V_{PQ}|Q\psi\rangle, \end{aligned} \quad (25)$$

which, upon left multiplication by  $\langle\phi_0|$ , yields

$$\langle\phi_0|P\psi\rangle = \frac{\langle\phi_0|V_{PQ}|Q\psi\rangle}{E - \bar{E}_0}. \quad (26)$$

In the second term on the right hand side of Eq. (25), the prime stands for the omission of the  $n = 0$  term in the spectral decomposition.

Next, the insertion of Eq. (25) into (24) leads to

$$|Q\psi\rangle = \frac{1}{E - h_{QQ}} V_{QP}|\phi_0\rangle \langle\phi_0|P\psi\rangle, \quad (27)$$

where the operator

$$h_{QQ} = H_{QQ} + V_{QP} \left(\frac{1}{E - \bar{\mathcal{H}}}\right)' V_{PQ} \quad (28)$$

has been introduced. Finally, by combining (26) and (27), one arrives at the equation

$$E - \bar{E}_0 = \langle\phi_0|V_{PQ} \frac{1}{E - h_{QQ}} V_{QP}|\phi_0\rangle, \quad (29)$$

which is the basis for computing the mean field energy error (or the fluctuations of the energy associated with randomness).

Although Eq. (29) is valid for any choice of the projectors  $P$  and  $Q$ , its use is in fact appropriate when the  $P$ -space is one-dimensional, as it was indeed

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<sup>a</sup>We omit here the difficult proof concerning the normalization and orthogonalization of the eigenfunctions  $|\phi_n\rangle$  of  $\bar{\mathcal{H}}$ .



the case in Ref. <sup>8</sup>, where this choice was made for sake of simplicity. For a two-dimensional  $P$ -space, as we shall later see, one should rather single out two, rather than one, terms in the spectral decomposition of the operator  $(E - \tilde{H})^{-1}$  on the right hand side of Eq. (25).

We refer the reader to Refs. <sup>6,7,8</sup> for a discussion on how the average of the square of Eq. (29) is actually computed (the average of (29) of course should vanish) and on how the complexity expansion is organized. Here, we confine ourselves to the leading term of this fast converging expansion.

## 2.2 Energy averaging

To understand better the significance of the energy averaging distribution (14) we show how it works in the simple cases of a bi-dimensional (A) and of a tri-dimensional (B) Hilbert space.

### A. Bi-dimensional Hilbert space

Let  $|\chi_1\rangle$  and  $|\chi_2\rangle$  be the two normalized states spanning the space. Here the only possible choice for the projectors clearly is

$$P \equiv |\chi_1\rangle\langle\chi_1| \quad \text{and} \quad Q \equiv |\chi_2\rangle\langle\chi_2|. \quad (30)$$

Then, by expanding the operator  $1/(E - H_{QQ})$ , Eq. (9) can be recast as follows

$$\left[ E - |\chi_1\rangle a_{11} \langle\chi_1| - |\chi_1\rangle a_{12} \langle\chi_2| \frac{1}{E} \sum_{n=0}^{\infty} \left( \frac{a_{22}}{E} \right)^n (|\chi_2\rangle\langle\chi_2|)^n |\chi_2\rangle a_{12}^* \langle\chi_1| \right] |P\psi\rangle = 0, \quad (31)$$

which, upon multiplying from the left by  $\langle\chi_1|$  and exploiting the idempotency of  $|\chi_2\rangle\langle\chi_2|$ , simplifies to

$$\left[ E - a_{11} - \frac{|a_{12}|^2}{E - a_{22}} \right] \langle\chi_1|P\psi\rangle = 0, \quad (32)$$

where the shorthand notations

$$a_{11} = \langle\chi_1|H|\chi_1\rangle, \quad a_{22} = \langle\chi_2|H|\chi_2\rangle \quad \text{and} \quad a_{12} = \langle\chi_1|H|\chi_2\rangle \quad (33)$$

have been introduced. This equation is trivially solved yielding the eigenvalues

$$E_{\pm} = \frac{1}{2} \left[ a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4|a_{12}|^2} \right], \quad (34)$$

which coincide with those of  $H$ . It helps notice that the eigenvalues (34) are also found as intersections of the hyperbola

$$E = a_{11} + \frac{|a_{12}|^2}{\omega - a_{22}} \quad (35)$$

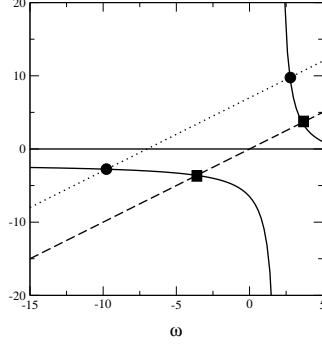


Figure 3. The eigenvalues of a bi-dimensional Hilbert space. The matrix elements of the Hamiltonian are taken to be  $a_{11} = -2$ ,  $a_{22} = 2$  and  $a_{12} = 3$ , in arbitrary units. The exact eigenvalues (squares) and the ones of the energy averaged Hamiltonian  $\bar{H}$  (circles) are shown. They correspond to the intersections with the straight line  $E = \omega + \epsilon$ : The former with  $\epsilon = 0$ , the latter with  $\epsilon = 7$ . The stability of the lowest eigenvalue and the upward shift of the highest one are clearly apparent.

with the straight line  $E = \omega$ .

Also the energy averaged Hamiltonian (20) can be expressed in the basis spanned by  $\chi_1$  and  $\chi_2$  and one gets the mean field equation

$$\left[ \bar{E} - a_{11} - \frac{|a_{12}|^2}{\bar{E} - \epsilon - a_{22}} \right] \langle \chi_1 | \langle P\psi \rangle \rangle = 0. \quad (36)$$

The latter is again trivially solved yielding

$$\bar{E}_{\pm} = \frac{1}{2} \left[ a_{11} + a_{22} + \epsilon \pm \sqrt{(a_{11} - a_{22} - \epsilon)^2 + 4|a_{12}|^2} \right], \quad (37)$$

which now corresponds to the intersections of the hyperbola (35) (with  $\bar{E}$  replacing  $E$ ) with the new straight line  $\bar{E} = \omega + \epsilon$ .

From Fig. 3, where the solutions  $E_{\pm}$  and  $\bar{E}_{\pm}$  are graphically displayed, it clearly appears that, while  $\bar{E}_{-} \cong E_{-}$ , the solution  $\bar{E}_{+}$  is much larger than  $E_{+}$ , the more so the greater  $\epsilon$  is. It is thus clear that the averaging distribution (14), while mildly affecting the eigenvalue of  $H$  lying in the  $P$ -space, drives away the one lying in the  $Q$ -space.

### B. Tri-dimensional Hilbert space

The space is spanned by the normalized states  $|\chi_1\rangle$ ,  $|\chi_2\rangle$  and  $|\chi_3\rangle$ . Now, two choices are possible for the projectors, namely

$$P \equiv |\chi_1\rangle\langle\chi_1| + |\chi_2\rangle\langle\chi_2| \quad (38)$$

$$Q \equiv |\chi_3\rangle\langle\chi_3| \quad (39)$$

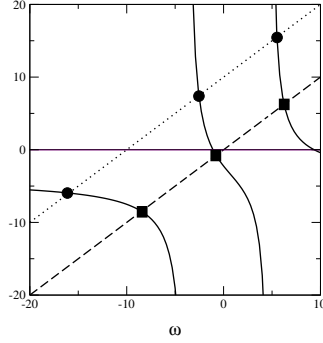


Figure 4. The eigenvalues of a tri-dimensional Hilbert space: The case of a one-dimensional  $P$ -space. The following matrix elements of the Hamiltonian are taken:  $a_{11} = -4$ ,  $a_{22} = -2$ ,  $a_{33} = 3$ ,  $a_{12} = 2$ ,  $a_{13} = -5$  and  $a_{23} = 3.5$ , in arbitrary units. The exact eigenvalues (squares) and the ones of the energy averaged Hamiltonian  $\bar{\mathcal{H}}$  (circles) are shown. They correspond to the intersections with the straight line  $E = \omega + \epsilon$ : The former with  $\epsilon = 0$ , the latter with  $\epsilon = 10$ . The stability of the  $P$ -space eigenvalue and the upward shift of those belonging to the  $Q$ -space are clearly apparent.

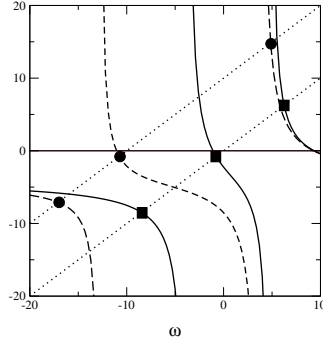


Figure 5. The eigenvalues of a tri-dimensional Hilbert space: The case of a bi-dimensional  $P$ -space. The matrix elements of the Hamiltonian are taken as in Fig. 4. The exact eigenvalues (squares) correspond to the intersections of the straight line  $E = \omega$  with the continuous curve. The eigenvalues of the energy averaged Hamiltonian  $\bar{\mathcal{H}}$  correspond to the intersections of the straight line  $\bar{E} = \omega + \epsilon$ , with  $\epsilon = 10$ , with the dashed curves. Note the dependence upon  $\epsilon$ . The stability of the  $P$ -space eigenvalue and the upward shift of the one belonging to the  $Q$ -space are again clearly apparent.

and

$$P \equiv |\chi_1\rangle\langle\chi_1| \quad (40)$$

$$Q \equiv |\chi_2\rangle\langle\chi_2| + |\chi_3\rangle\langle\chi_3|. \quad (41)$$

In both cases, Eq. (9) can be recast as follows

$$(E - a_{11})(E - a_{22})(E - a_{33}) - |a_{12}|^2(E - a_{33}) - |a_{13}|^2(E - a_{22}) - |a_{23}|^2(E - a_{11}) = 0, \quad (42)$$

which is the cubic equation yielding the exact eigenvalues. Note that Eq. (42) is easily obtained with the choice (2.2), because in this case the operator  $(E - H_{QQ})^{-1}$  is expanded in terms of the idempotent operator  $|\chi_3\rangle\langle\chi_3|$ . Not so with the choice (2.2), because now  $(E - H_{QQ})^{-1}$  should be expanded in terms of the operator (41), *which is not idempotent*. Actually, the larger the powers of the latter are, the more cumbersome they become. Yet, also in this case it can be proved that Eq. (42) holds valid.

Let us now examine the solutions of Eq. (20): As in the previous bi-dimensional case it is convenient to display the solutions graphically. For the partition (2.2), one finds that they are given by the intersections of the  $\epsilon$ -independent curve

$$\bar{E} = a_{11} + \frac{|a_{12}|^2(\omega - a_{33})}{\mathcal{D}_1(\omega)} + \frac{|a_{13}|^2(\omega - a_{22})}{\mathcal{D}(\omega)} + \frac{a_{12}a_{13}^*a_{23}}{\mathcal{D}(\omega)} + \frac{a_{12}^*a_{13}a_{23}^*}{\mathcal{D}(\omega)}, \quad (43)$$

where

$$\mathcal{D}(\omega) = (\omega - a_{22})(\omega - a_{33}) - |a_{23}|^2, \quad (44)$$

with the straight line  $\bar{E} = \omega + \epsilon$ , as displayed in Fig. 4, where the case  $\epsilon = 0$ , — which clearly provides the exact eigenvalues  $E_i$  of the Schroedinger equation, — is also shown. From the figure, it transparently appears that  $\bar{E}_0 \cong E_0$ , whereas  $\bar{E}_1 \gg E_1$  and  $\bar{E}_2 \gg E_2$ , the latter inequalities being stronger when the parameter  $\epsilon$  is large.

In the case of the partition (2.2), the solutions are given by the intersections of the curve obtained replacing  $a_{22} \rightarrow a_{22} - \epsilon$  in Eqs. (43) and (44) with the straight line  $\bar{E} = \omega + \epsilon$ , as displayed in Fig. 5. We face here a new situation, since now not only the straight line, but also Eqs. (43) and (44) are  $\epsilon$ -dependent. Yet, one again sees that for  $\epsilon = 0$  one recovers the eigenvalues  $E_i$ , whereas when  $\epsilon \neq 0$  the intercepts occur for  $\bar{E}_0 \cong E_0$  and  $\bar{E}_1 \cong E_1$ , but for  $\bar{E}_2 \gg E_2$ . Hence, we conclude that the action of the averaging distribution (14) affects very little the eigenvalues belonging to the  $P$ -space, while pushing off the ones in the  $Q$ -space by an amount proportional to  $\epsilon$ .

### 2.3 The $P$ -space

Having defined the energy averaging procedure, to get further it is necessary to define the operators  $P$  and  $Q$ . For this purpose the natural candidates as building blocks of the  $P$  operator appear to be the eigenstates  $|\phi_n\rangle$  of the mean field Hamiltonian (20), defined by the equation

$$\bar{\mathcal{H}}|\phi_n\rangle = \bar{E}_n|\phi_n\rangle. \quad (45)$$

Their finding requires, however, the solution of a difficult self-consistency problem. Hence, we make the simpler choice of viewing as building blocks of  $P$  the Hartree-Fock (HF) variational solutions, which are, e. g., trivial in nuclear matter, the system we shall consider.

For sake of illustration we start with a one-dimensional  $P$ -space by setting

$$P = |\chi_{\text{HF}}\rangle\langle\chi_{\text{HF}}|, \quad (46)$$

$|\chi_{\text{HF}}\rangle$  being the HF ground state wave function of nuclear matter (the Fermi sphere). Then, on the basis of (46), one derives the mean field equation

$$\bar{E}_0 = E_{\text{HF}} + \frac{\beta^2}{E_0 - \epsilon - E}, \quad (47)$$

which relates the mean field ( $\bar{E}_0$ ), the HF ( $E_{\text{HF}}$ ) and the true ( $E$ ) energies *per particle*, and the equation for the statistical fluctuation of the energy

$$E - \bar{E}_0 = \pm \frac{1}{E - \bar{\epsilon}_2} \sqrt{\frac{2}{N_2}} \beta^2, \quad (48)$$

where

$$\beta^2 = \sum_{2\text{p}-2\text{h}} |\langle\psi_{2\text{p}-2\text{h}}|V|\chi_{\text{HF}}\rangle|^2, \quad (49)$$

the bras  $\langle\psi_{2\text{p}-2\text{h}}|$  representing the two-particle–two-holes (2p-2h) states of nuclear matter, whose average energy *per particle* is  $\bar{\epsilon}_2$ . Thus all the quantities appearing in Eqs. (47) and (48) are *per particle*, including the parameter  $\epsilon$  and the residual effective interaction  $V$  (which are accordingly divided by the nuclear mass number  $A$ ).

Note also that Eq. (48) gives the “fluctuations” of the mean field energy in the first order of the complexity expansion, which is based on an organization of the  $Q$ -space in blocks of excited states of increasing complexity (see Fig. 6): Here, the contribution to the error only arises from the sector of the  $Q$ -space set up with the 2p-2h excitations. Moreover, although the states of the  $Q$ -space obey well-defined, coupled differential equations (see Ref. <sup>7</sup>), we describe

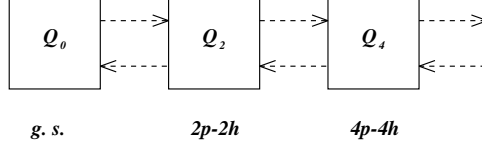


Figure 6. The partition of the Hilbert space of nuclear matter in sets of increasing complexity. The first box on the left defines the  $P$ -space, the second one embodies the simplest states in the  $Q$ -space and so on.

them with the HF multi-particle-multi-hole solutions, an approximation not impairing the orthogonality constraint  $P \cdot Q = 0$ .

The sum in Eq. (49) is performed over the ensemble of the 2p-2h excited states lying in an appropriate energy range (in Refs. <sup>7,8</sup> taken to be fixed essentially by the parameter  $\epsilon$ ), whose number  $\mathcal{N}_2$  can be computed using the Ericson's formula<sup>11</sup> for the density of the spin  $J$   $N$ -particle- $N$ -hole nuclear states, namely

$$\rho_{ph}^{(N)}(\mathcal{E}, J) = \frac{g(g\mathcal{E})^{N-1}}{p!h!(N-1)!} \frac{2J+1}{\sqrt{8\pi}\sigma^3 N^{3/2}} \exp[-(2J+1)^2/(8N\sigma^2)], \quad (50)$$

where

$$g = \frac{3}{2} \frac{A}{\epsilon_F} \quad \text{and} \quad \sigma^2 = \mathcal{F} \sqrt{\frac{\mathcal{E}}{a}} \frac{1}{\hbar^2}, \quad (51)$$

$\mathcal{F}$  being the nuclear moment of inertia,  $\epsilon_F$  the Fermi energy,  $a = A/8 \text{ MeV}^{-1}$  and  $\mathcal{E}$  the excitation energy of the system.

Now, Eqs. (47) and (48), owing to the double sign appearing in the latter, set up two systems, each one including two equations, in two unknowns. Two options are then possible in selecting the latter: One can choose either the ground state mean field and true energies per particle, i. e.  $\bar{E}_0$  and  $E$ , — assuming the matrix elements of the residual effective interaction to be known, — or  $\bar{E}_0$  and  $\beta^2$ , when  $E$  is experimentally known, — which is indeed the case in nuclear matter.

We take the latter choice, requiring in addition the coincidence of the two  $\bar{E}_0$  obtained by solving the two systems separately. Actually, and notably, both systems lead to the same formal expression for the mean field energy per particle, namely

$$\bar{E}_0 = \frac{1}{2} \left\{ (E_{\text{HF}} + E + \epsilon) - \sqrt{(E_{\text{HF}} - E - \epsilon)^2 + 4\beta^2} \right\}, \quad (52)$$

which holds valid for  $\epsilon < E_{\text{HF}} - E$ , the right hand side of this inequality being positive because of the variational principle. The above, when  $\beta^2 \rightarrow 0$ , yields

$$\bar{E}_0 = E + \epsilon, \quad (53)$$

in accord with (21) and (22), but also with (48), which, for  $V \rightarrow 0$ , gives  $\bar{E}_0 = E$ : Indeed, when the residual effective interaction vanishes no fluctuations can occur and hence the parameter  $\epsilon$  should vanish as well.

On the other hand, the two systems yield two different expressions for the sum of the matrix elements of the residual effective interaction squared, namely

$$\beta_l^2 = \sqrt{\mathcal{N}_2/2} \frac{E_l - \bar{\epsilon}_2}{2} \left\{ \left[ E_l(1 + \sqrt{\mathcal{N}_2/2}) - (\epsilon + \sqrt{\mathcal{N}_2/2} \bar{\epsilon}_2) - E_{\text{HF}} \right] \right. \\ \left. + \sqrt{\left[ E_l(1 + \sqrt{\mathcal{N}_2/2}) - (\epsilon + \sqrt{\mathcal{N}_2/2} \bar{\epsilon}_2) - E_{\text{HF}} \right]^2 + 4\epsilon(E_l - E_{\text{HF}})} \right\} \quad (54)$$

and

$$\beta_u^2 = \sqrt{\mathcal{N}_2/2} \frac{E_u - \bar{\epsilon}_2}{2} \left\{ \left[ E_u(-1 + \sqrt{\mathcal{N}_2/2}) - (-\epsilon + \sqrt{\mathcal{N}_2/2} \bar{\epsilon}_2) + E_{\text{HF}} \right] \right. \\ \left. + \sqrt{\left[ E_u(-1 + \sqrt{\mathcal{N}_2/2}) - (-\epsilon + \sqrt{\mathcal{N}_2/2} \bar{\epsilon}_2) + E_{\text{HF}} \right]^2 + 4\epsilon(E_u - E_{\text{HF}})} \right\}, \quad (55)$$

both of which vanish in the limit  $\epsilon \rightarrow 0$ , in accord with the previous discussion.

Formula (54), solution of the first system of equations (the one with the “+” sign on the right hand side of (48)), yields the value of the *energy dependent* quantity (49) (here denoted by  $\beta_l^2$ ) on the lower border of the energy band expressing the fluctuations of the ground state energy  $E$  (and thus encompassing  $\bar{E}_0$ ). Formula (55), solution of the second system of equations (the one with the minus sign on the right hand side of (48)), provides instead (49) (here denoted by  $\beta_u^2$ ) on the upper border of the band (remember that  $E - \bar{\epsilon}_2 < 0$ ).

Of course, of the energy per particle  $E$  we only know the experimental value, not the values on the borders of the band: As a consequence, we can only surmise the width  $W$  of the latter, thus providing two different inputs for the energy  $E$  appearing on the right hand side of (2.3), namely  $E_u = E + W/2$  and  $E_l = E - W/2$ . However, we can explore whether, for a given  $W$ , a value for the parameter  $\epsilon$  can be found (not too large, not too small) such to have the two mean field energies per particle to coincide. If this search succeeds, then an orientation on  $W$  (or, equivalently, on the size of the fluctuations of the ground state energy) can be gained.

Note that the framework above outlined holds because the same quantity  $\beta^2$  appears in both (47) and (48). This occurrence stems from an approximation whose validity is discussed in Ref. <sup>8</sup>.

We now extend the formalism by letting the projector  $P$  to encompass, beyond the ground state, the 2p-2h excitations of the HF variational scheme as well. Thus, instead of Eq. (46), we write

$$P = |\chi_{\text{HF}}\rangle\langle\chi_{\text{HF}}| + \sum_{\beta} |\chi_{\text{HF}}^{2\beta}\rangle\langle\chi_{\text{HF}}^{2\beta}|, \quad (56)$$

where the sum is meant to be extended to the whole set of 2p-2h HF excitations  $|\chi_{\text{HF}}^{2\beta}\rangle$ .

With the choice (56) the mean field Hamiltonian (20) is then defined and one can compute the mean field ground state energy per particle,  $\bar{E}_0 = \langle\phi_0|\mathcal{H}|\phi_0\rangle$ , using for the ket  $|\phi_0\rangle$  the expression

$$|\phi_0\rangle = s_0|\chi_{\text{HF}}\rangle + \sum_{\gamma} s_2^{\gamma}|\chi_{\text{HF}}^{2\gamma}\rangle, \quad (57)$$

and accounting for the influence of the  $Q$ -space on the ground state mean energy per particle in first order of the complexity expansion, i. e. by setting

$$Q = \sum_{\gamma} |\chi_{\text{HF}}^{4\gamma}\rangle\langle\chi_{\text{HF}}^{4\gamma}|, \quad (58)$$

the sum running over the whole set of the HF 4p-4h excitations. In (57)  $s_0$  and  $s_2^{\gamma}$  are complex coefficients, fixed, in principle, by Eq. (45) and satisfying the normalization condition

$$|s_0|^2 + \sum_{\gamma} |s_2^{\gamma}|^2 = 1. \quad (59)$$

After straightforward, but lengthy, algebra using (57) and (58) one arrives at the following new mean field equation

$$\begin{aligned} \bar{E}_0 = & |s_0|^2 E_{\text{HF}} + \sum_{\gamma} |s_2^{\gamma}|^2 \langle\chi_{\text{HF}}^{2\gamma}|H|\chi_{\text{HF}}^{2\gamma}\rangle + 2s_0^* \sum_{\gamma} s_2^{\gamma} \langle\chi_{\text{HF}}|\mathcal{V}|\chi_{\text{HF}}^{2\gamma}\rangle + \\ & + \frac{1}{\bar{E}_0 - \epsilon - E} \sum_{\beta\gamma} |s_2^{\gamma}|^2 |\langle\chi_{\text{HF}}^{4\beta}|\mathcal{V}|\chi_{\text{HF}}^{2\gamma}\rangle|^2, \end{aligned} \quad (60)$$

$\mathcal{V}$  being the bare NN potential.

Notably, Eq. (60) turns out to formally coincide with (47). Indeed, the first three terms on the right hand side of Eq. (60) just yield the mean value of the original bare Hamiltonian  $H$  in the state (57). In the thermodynamic limit of nuclear matter out of these pieces only the HF energy survives. Indeed, the



correction to the HF energy per particle due to a finite number of particle-hole excitations vanishes in the thermodynamic limit. In other words, the 2p-2h admixture into (57) *does not change* the expectation value of the Hamiltonian.

Hence, defining

$$\zeta^2 = \sum_{\beta\gamma} |s_2^\gamma|^2 |\langle \chi_{\text{HF}}^{4\beta} | V | \chi_{\text{HF}}^{2\gamma} \rangle|^2, \quad (61)$$

(60) can be recast into the form

$$\bar{E}_0 = E_{\text{HF}} + \frac{\zeta^2}{\bar{E}_0 - \epsilon - E}, \quad (62)$$

whose similarity with Eq. (47) is transparent.

The same will take place for *any* admixture of  $N_p$ - $N_h$  HF excited states in  $|\chi_{\text{HF}}\rangle$ : Hence, in our framework different choices of the projection operator  $P$  lead to the same structure for the mean field equation for nuclear matter. This invariance does not hold in finite nuclei.

The only, of course important, difference between (49) and (61) relates to the residual interaction  $V$ , which in (61) induces transitions from 2p-2h to 4p-4h states, rather than from the Fermi sphere to the 2p-2h states.

Concerning the statistical fluctuation equation one can again use (29), with the state  $|\phi_0\rangle$  given now by Eq. (57). Then invoking *the randomness of the phases of the  $Q$ -space wave functions (RPA)* and proceeding exactly as done in Refs. <sup>7,8</sup>, one deduces the new fluctuation equation

$$E - \bar{E}_0 = \pm \frac{1}{E - \bar{\epsilon}_4} \sqrt{\frac{2}{\mathcal{N}_4}} \zeta^2, \quad (63)$$

where  $\bar{\epsilon}_4$  denotes the average energy per particle of the 4p-4h HF states. In (63),  $\mathcal{N}_4$  represents the number of 4p-4h excitations contributing to the sum over the index  $\beta$  in (61).

Hence, the “formal invariance”, with respect to the choice for the projector  $P$ , holds for both the equations at the core of our statistical approach to nuclear matter in first order of the complexity expansion.

Indeed, the inclusion of  $N_p$ - $N_h$  states (with  $N > 2$ ) into the  $P$ -space would merely imply the replacement, in (63), of  $\mathcal{N}_4$  with  $\mathcal{N}_{N+2}$  and, at the same time, to have  $\zeta^2$  defined in terms of the matrix elements of  $V$  between  $N_p$ - $N_h$  and  $(N+2)p$ -( $N+2$ )h states. In addition, one should of course insert in the energy denominator the average energy per particle of the  $(N+2)p$ -( $N+2$ )h HF states.

Therefore, the extension of the  $P$ -space rapidly leads to the vanishing of the fluctuations, owing to the very fast increase of the number  $\mathcal{N}_N$ .

However, as already mentioned, Eq. (29) *is not* in general a good starting point to derive the fluctuation equation. Indeed, it selects out only *one term* in the spectral decomposition of the operator  $1/(E - \bar{\mathcal{H}})$ , which is appropriate for a one-dimensional  $P$ -space only. Hence, in place of (25), we rather write

$$|P\psi\rangle = |\phi_0\rangle \frac{\langle\phi_0|V_{PQ}|Q\psi\rangle}{E - \bar{E}_0} + |\phi_2\rangle \frac{\langle\phi_2|V_{PQ}|Q\psi\rangle}{E - \bar{E}_2} + \left(\frac{1}{E - \bar{\mathcal{H}}}\right)'' V_{PQ}|Q\psi\rangle, \quad (64)$$

with an obvious meaning of the double primed operator in the last term on the right hand side. Clearly, Eq. (64) does not follow directly from (56), but it assumes that in the spectral decomposition of the  $(E - \bar{\mathcal{H}})^{-1}$  operator only one prominent (collective) state  $|\phi_2\rangle$  enters beyond the ground state  $|\phi_0\rangle$ .

Sticking to this model, instead of Eq. (27), we likewise write

$$|Q\psi\rangle = \frac{1}{E - h_{QQ}^{(2)}} V_{QP} \{|\phi_0\rangle \langle\phi_0|P\psi\rangle + |\phi_2\rangle \langle\phi_2|P\psi\rangle\}, \quad (65)$$

being

$$h_{QQ}^{(2)} = H_{QQ} + V_{QP} \left(\frac{1}{E - \bar{\mathcal{H}}}\right)'' V_{PQ}. \quad (66)$$

Then, by left multiplying (64) (with  $E = E_0$ ) by  $\langle\phi_0|$  and using (65), we obtain

$$E_0 - \bar{E}_0 = \langle\phi_0|V_{PQ} \frac{1}{E_0 - h_{QQ}^{(2)}} V_{QP}|\phi_0\rangle + \langle\phi_0|V_{PQ} \frac{1}{E_0 - h_{QQ}^{(2)}} V_{QP}|\phi_2\rangle \frac{\langle\phi_2|P\psi\rangle}{\langle\phi_0|P\psi\rangle}, \quad (67)$$

which generalizes Eq. (29).

In a perfectly analogous fashion, by left multiplying (64) (with  $E = E_2$ ) by  $\langle\phi_2|$  and using (65), we obtain

$$E_2 - \bar{E}_2 = \langle\phi_2|V_{PQ} \frac{1}{E_2 - h_{QQ}^{(2)}} V_{QP}|\phi_2\rangle + \langle\phi_2|V_{PQ} \frac{1}{E_2 - h_{QQ}^{(2)}} V_{QP}|\phi_0\rangle \frac{\langle\phi_0|P\psi\rangle}{\langle\phi_2|P\psi\rangle}. \quad (68)$$

In the above,  $E_0$  and  $E_2$  stand for the first two exact eigenvalues of the Schroedinger equation;  $\bar{E}_0$  and  $\bar{E}_2$  for the corresponding quantities associated with Eq. (45). Eq. (68) shows that, in the present framework, all the energies of the  $P$ -space fluctuate.

Now, the energy averaging of (67) and (68) vanishes by definition, but the energy averaging of their square, which yields the “error”, does not. Hence, proceeding along the lines of Refs. <sup>7,8</sup>, we subtract on the right hand side of both equations their average values, square the expressions thus obtained and

make use of RPA, keeping of our expansion in the complexity of the  $Q$ -space states the first term only. Next, we exploit the structure of  $|\phi_2\rangle$ , which, like  $|\phi_0\rangle$ , must be normalized, orthogonal to  $|\phi_0\rangle$  and of the form

$$|\phi_2\rangle = \sum_{\beta} c_2^{\beta} |\chi_{\text{HF}}^{2\beta}\rangle + c_0 |\chi_{\text{HF}}\rangle. \quad (69)$$

Finally, we arrive at the equation

$$E_0 - \bar{E}_0 = \pm \frac{1}{E_0 - \bar{\epsilon}_4} \sqrt{\frac{2}{\mathcal{N}_4}} (\zeta^2 + r \xi^2), \quad (70)$$

where, in addition to (61), the further definition

$$\xi^2 = \sum_{\beta\gamma} s_2^{\gamma*} c_2^{\gamma} |\langle \chi_{\text{HF}}^{4\beta} | V | \chi_{\text{HF}}^{2\gamma} \rangle|^2 \quad (71)$$

has been introduced; moreover, we have set

$$r \equiv \frac{\langle \phi_2 | P \psi \rangle}{\langle \phi_0 | P \psi \rangle}. \quad (72)$$

Likewise, for the energy of the 2p-2h state of the  $P$ -space one obtains the fluctuation equation

$$E_2 - \bar{E}_2 = \pm \frac{1}{E_2 - \bar{\epsilon}_4} \sqrt{\frac{2}{\mathcal{N}_4}} (\eta^2 + \frac{\xi^2}{r}), \quad (73)$$

where, naturally,

$$\eta^2 = \sum_{\beta\gamma} |c_2^{\gamma}|^2 |\langle \chi_{\text{HF}}^{4\beta} | V | \chi_{\text{HF}}^{2\gamma} \rangle|^2. \quad (74)$$

We thus see that the statistical fluctuation equations (70) and (73) are actually coupled through the term (71).

Concerning the mean field equations, clearly with the projector (56) an equation should exist also for the energy of the 2p-2h state. It can be derived by computing  $\bar{E}_2 = \langle \phi_2 | \bar{\mathcal{H}} | \phi_2 \rangle$  and, notably, it turns out to read

$$\bar{E}_2 = E_{\text{HF}}^{(2)} + \frac{\eta^2}{\bar{E}_2 - \epsilon - E_2}, \quad (75)$$

$E_{\text{HF}}^{(2)}$  representing the HF energy per particle of the system in the 2p-2h excited state. Since we split  $E_{\text{HF}}^{(2)}$  into a part associated with the HF ground state and a part associated with the 2p-2h *excitation energies*, both per particle, and since the latter vanishes in the thermodynamic limit, — as previously

noted in commenting Eq. (60), then  $E_{\text{HF}}^{(2)} = E_{\text{HF}}$ , a relation we expect to be approximately fulfilled also in a heavy nucleus.

We conclude from the above analysis that our approach leads to a set of mean field equations, one for each of the states lying in the  $P$ -space: These equations, unlike the fluctuation ones, are not coupled.

#### 2.4 Normalization and fluctuation of the $P$ -space ground state wave function

In the present framework the ground state spectroscopic factor  $S$  is the square root of the norm of  $|P\psi\rangle$ , the system's ground state wave function projection in  $P$ -space. To find it we exploit the completeness of the normalized eigenstates of  $\mathcal{H}$ . Hence we write

$$S^2 \equiv \langle P\psi|P\psi\rangle = \sum_{n=0}^M \langle P\psi|\phi_n\rangle \langle \phi_n|P\psi\rangle = 1 - \langle Q\psi|Q\psi\rangle. \quad (76)$$

Now, confining ourselves to set  $M = 1$ , then the Eq. (27) for  $|Q\psi\rangle$  is warranted and we rewrite (76) as follows

$$S^2 = 1 - \langle \phi_0|V_{PQ} \frac{1}{(E_0 - h_{QQ})^2} V_{QP}|\phi_0\rangle |\langle \phi_0|P\psi\rangle|^2. \quad (77)$$

Moreover, when  $M = 1$  then

$$S^2 = |\langle \phi_0|P\psi\rangle|^2. \quad (78)$$

Hence, by exploiting (29), Eq. (77) can be recast into the form<sup>7,8</sup>

$$S^2 = 1 + S^2 \left[ \frac{d}{dE_0}(E_0 - \bar{E}_0) + \frac{E_0 - \bar{E}_0}{\bar{E}_0 - \epsilon - E_0} \right]. \quad (79)$$

Finally, employing (52) the expression

$$S^2 = \left[ \frac{3}{2} + \frac{1}{2} \frac{E_{\text{HF}} - E_0 - \epsilon - 2d\beta^2/dE_0}{\sqrt{(E_{\text{HF}} - E_0 - \epsilon)^2 + 4\beta^2}} + \frac{\epsilon}{\bar{E}_0 - \epsilon - E_0} \right]^{-1} \quad (80)$$

follows<sup>7,8</sup>, where the energy derivative of the sum of the square moduls of the vacuum-2p-2h matrix elements of the effective interaction appears (its explicit expression is given in Ref. <sup>8</sup>). Note that (80) goes to one as  $\beta^2 \rightarrow 0$ , as it should.

If, however, the expression for the  $Q$ -space wave function appropriate for a two-dimensional  $P$ -space, namely (65), is used, then (see Ref. <sup>12</sup> for details)

one ends up with the expression

$$S^2 = \frac{1 - \left[ \frac{d(\bar{E}_2 - \bar{E}_0)}{dE_0} + \frac{\bar{E}_2 - \bar{E}_0}{E_0 - \epsilon - E_0} \right] |\langle \phi_2 | P\psi \rangle|^2}{1 + \frac{\epsilon}{E_0 - \epsilon - E_0} + \frac{d\bar{E}_0}{dE_0}}, \quad (81)$$

which reduces to (79), as it should, if  $\langle \phi_2 | P\psi \rangle \rightarrow 0$ , i. e. for a one-dimensional  $P$ -space. In deducing (81) the approximation

$$\frac{1}{E_0 - h_{QQ}^{(2)}} \approx \frac{1}{E_2 - h_{QQ}^{(2)}} \quad (82)$$

has been made.

Since, from Eq. (72),

$$|\langle \phi_2 | P\psi \rangle|^2 = \frac{r^2}{1 + r^2} S^2, \quad (83)$$

then Eq. (81) can be recast as follows

$$S^2 = \left\{ 1 + \frac{1}{\bar{E}_0 - \epsilon - E_0} \left[ \epsilon + \frac{r^2}{1 + r^2} (\bar{E}_2 - \bar{E}_0) \right] + \frac{1}{1 + r^2} \left( \frac{d\bar{E}_0}{dE_0} + r^2 \frac{d\bar{E}_2}{dE_0} \right) \right\}^{-1}, \quad (84)$$

which again reduces to (79) as  $r \rightarrow 0$ .

We now address the problem of the fluctuation of  $|P\psi\rangle$ . For this scope, we focus on the ground state and, by combining Eqs. (17), (23) and (24), we obtain

$$(E - \bar{\mathcal{H}})[|P\psi\rangle - |\langle P\psi \rangle\rangle] + (E - \bar{E}_0)|\langle P\psi \rangle\rangle = V_{PQ}|Q\psi\rangle \quad (85)$$

(the angle brackets meaning energy averaging).

Then, if use is made of the expression (27) for  $|Q\psi\rangle$  and of the spectral decomposition of the operator  $(E - \bar{\mathcal{H}})^{-1}$ , one gets

$$\begin{aligned} |P\psi\rangle - |\langle P\psi \rangle\rangle &= \sum_n' |\phi_n\rangle \frac{1}{E - \bar{E}_n} \langle \phi_n | V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} | \phi_0 \rangle \langle \phi_0 | P\psi \rangle \\ &\quad + \frac{1}{E - \bar{E}_0} |\phi_0\rangle \langle \phi_0 | V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} | \phi_0 \rangle \langle \phi_0 | P\psi \rangle \\ &\quad - (E - \bar{E}_0) \sum_n \frac{1}{E - \bar{E}_n} |\phi_n\rangle \langle \phi_n | \langle P\psi \rangle\rangle. \end{aligned} \quad (86)$$

Now, since  $|\langle P\psi \rangle\rangle \propto |\phi_0\rangle$ , from the above finally it follows

$$|P\psi\rangle - |\langle P\psi \rangle\rangle = (1 - |\phi_0\rangle \langle \phi_0|)^{-1} \left( \frac{1}{E - \bar{\mathcal{H}}} \right)' V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} | \phi_0 \rangle \langle \phi_0 | P\psi \rangle$$

$$= \langle \phi_0 | P\psi \rangle \left( \frac{1}{E_0 - \mathcal{H}} \right)' V_{PQ} \frac{1}{E_0 - h_{QQ}} V_{QP} | \phi_0 \rangle, \quad (87)$$

which *vanishes when  $P$  is given by (46)*, since clearly  $|P\psi\rangle$  does not fluctuate in a one-dimensional  $P$ -space.

If, on the other hand,  $P$  is given by Eq. (56), then the above can be computed in first order of the complexity expansion, using (57) for  $|\phi_0\rangle$ , (69) for  $|\phi_2\rangle$  and (58) for  $Q$ . One ends up with the expression (we set  $E = E_0$  to conform to previous notations)

$$[|P\psi\rangle - \langle P\psi \rangle]_1 = \langle \phi_0 | P\psi \rangle \frac{|\phi_2\rangle}{E_0 - E_2} \sum_{\beta\beta'\gamma} \langle \chi_{\text{HF}}^{2\beta} | V | \chi_{\text{HF}}^{4\gamma} \rangle \frac{c_{2\beta}^* s_{2\beta'}}{E_0 - \epsilon_{\text{HF}}^{4\gamma}} \langle \chi_{\text{HF}}^{4\gamma} | V | \chi_{\text{HF}}^{2\beta'} \rangle, \quad (88)$$

which can be further simplified invoking the randomness of the phases of the wave functions in the  $Q$ -space and again introducing the average energy per particle  $\bar{\epsilon}_4$  for the 4p-4h HF excited states. Then, with the help of Eq. (71), the formula

$$[|P\psi\rangle - \langle P\psi \rangle]_1 = \frac{\langle \phi_0 | P\psi \rangle}{(E_0 - E_2)(E_0 - \bar{\epsilon}_4)} |\phi_2\rangle \xi^2 \quad (89)$$

is derived. It gives the fluctuations of the wave function in first order of the complexity expansion.

Notice that in (89) the scalar product  $\langle \phi_0 | P\psi \rangle$ , unlike in Refs. <sup>7,8</sup>, does not coincide with the spectroscopic factor  $S$ , as defined in (76), since Eq. (89) refers to a  $P$ -space with dimensions larger than one. Rather, it measures the amount of the true ground state wave function of the system embodied in the mean field state  $|\phi_0\rangle$ .

The relevance of Eq. (89) lies in the possibility it offers to assess the “error” affecting the ground state expectation value of operators associated to physical observables like, e. g., the magnetic moments of nuclei.

### 3 A glance at QCD

Lately the existence of randomness has been unraveled also at the scale of quarks, namely at the level of the physics ruled by QCD. Here the RMT has achieved outstanding successes in connection with the low-lying eigenvalues of the Dirac operator for massless fermions:

$$iD = i\cancel{D} + g \sum_a \frac{\lambda^a}{2} A^a, \quad (90)$$

where  $g$  is the coupling constant, the matrices  $\lambda^a$  the generators of the gauge group and the  $A^a$  the gauge fields. Indeed, the spectrum of (90),

$$iD\phi_n = \lambda_n\phi_n, \quad (91)$$

has been computed for a SU(2) gauge theory (and for others as well) and from the  $\lambda_n$ 's the nearest-neighbor spacing distribution  $P(s)$  has been deduced. In turn, this has been compared to the prediction of the chRMT (chiral random matrix theory): An impressive agreement has been found,<sup>13</sup> pointing to the universality of the fluctuations of the eigenvalues of the Dirac operator.

The focus on the  $\lambda_n$ 's also stems from the anticommutator

$$\{iD, \gamma_5\} = 0, \quad (92)$$

whose vanishing in the chiral limit implies for the eigenvalues of (91) the occurrence in symmetrical pairs  $(\pm\lambda_n)$  around  $\lambda = 0$ . As a consequence the  $\lambda_n$ 's tend to accumulate around the origin and their average density

$$\rho(\lambda) = \langle \sum_n \delta(\lambda - \lambda_n) \rangle, \quad (93)$$

the average being taken on all the configurations of the gauge field, on the one hand displays the level repulsion already encountered at the level of the physics of the nucleus, — thus signalling universality, — and on the other plays a central role in the physics of the quark-gluon plasma.

In fact, the density of the smallest eigenvalues of the Dirac operator is directly related to the chiral condensate according to Banks–Casher formula<sup>14</sup>:

$$\langle \bar{q}q \rangle = -\pi\rho(0)/V, \quad (94)$$

$V$  being the space-time volume. It is fascinating that such a fundamental aspect of nature as the breaking of the chiral symmetry appears to be ruled by chaos.

In conclusion, we feel that RMT, in both its normal and chiral versions, is invaluable in disentangling the stochastic content of nuclear physics and QCD. In the former case, in the Feshbach's approach, stochasticity should be averaged out, thus leading to an effective Hamiltonian that has been proved most successful in interpreting the data. We argue that the same path might be worth following also in QCD.

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